

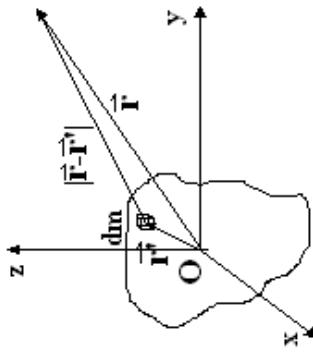
Some topics on compact objects and the spacetime around them

George Pappas

TAT Tuesday seminar

October 30, 2012

- The gravitational field of a mass configuration and a few words on multipole moments.
- Compact objects as fluid configurations.
- Spacetimes for rapidly rotating neutron stars.
- Orbits in the spacetime around compact objects.
- Frequencies of the orbits around compact objects.
- Multipole moments from orbits.



$$\Phi(r) = G \int \frac{\rho(r') dV'}{|\vec{r} - \vec{r}'|}$$

Multipolar Expansions

In spherical coordinates:

$$\begin{aligned} \frac{1}{|\vec{r} - \vec{r}'|} &= \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta'}} = \frac{1}{r\sqrt{1 + \varepsilon^2 - 2\varepsilon \cos \theta'}} \\ &= \frac{1}{r} + \frac{\cos \theta' \varepsilon}{r} + \frac{(1 + 3 \cos 2\theta') \varepsilon^2}{4r} + \dots \end{aligned}$$

$$\Phi(r) = G \left(\frac{1}{r} \int \rho(r') dV' + \frac{1}{r^2} \int r' \cos \theta' \rho(r') dV' + \frac{1}{r^3} \int \frac{1}{2} (3 \cos^2 \theta' - 1) (r')^2 \rho(r') dV' + \dots \right)$$

In cartesian coordinates:

$$\begin{aligned} \frac{1}{|\vec{r} - \vec{r}'|} &= \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} = \frac{1}{r \sqrt{(x^a/r - \varepsilon^a)(x^b/r - \varepsilon^b) \delta_{ab}}} \\ &= \frac{1}{r} + \frac{x^a \varepsilon_a}{r^2} + \frac{\frac{1}{2}(3x^a x^b - r^2 \delta^{ab}) \varepsilon_a \varepsilon_b}{r^3} + \dots = \frac{1}{r} + \frac{x^a x'_a}{r^3} + \frac{\frac{1}{2}(3x'_a x'_b - r'^2 \delta_{ab}) x^a x^b}{r^5} + \dots \end{aligned}$$

$$\Phi(r) = G \left(\frac{1}{r} \int \rho(r') dV' + \frac{x^a}{r^3} \int x'_a \rho(r') dV' + \frac{x^a x^b}{r^5} \int \frac{1}{2} (3x'_a x'_b - r'^2 \delta_{ab}) \rho(r') dV' + \dots \right)$$

Newtonian multipole moments:

$$\Phi(r) = G \left(\frac{\mathcal{Q}}{r} + \frac{\mathcal{Q}_a x^a}{r^3} + \frac{\mathcal{Q}_{ab} x^a x^b}{r^5} + \dots \right) \quad (1)$$

where, \mathcal{Q} , \mathcal{Q}_a , \mathcal{Q}_{ab} , are some integrals on the source

$$\mathcal{Q} = \int \rho(r') d^3x', \quad \mathcal{Q}_a = \int x'_a \rho(r') d^3x', \quad \mathcal{Q}_{ab} = \int \frac{3}{2}(x'_a x'_b - \frac{1}{3} r'^2 \delta_{ab}) \rho(r') d^3x' \dots \quad (2)$$

The multipole moments are generally tensorial quantities.

Definition of the moments at infinity:

$$x^a \rightarrow \tilde{x}^a = r^{-2} x^a; \quad \tilde{r}^2 = \tilde{x}^a \tilde{x}_a = r^{-2}$$

$$\Phi(r) = \tilde{r} (\mathcal{Q} + \mathcal{Q}_a \tilde{x}^a + \mathcal{Q}_{ab} \tilde{x}^a \tilde{x}^b + \dots) \quad (3)$$

If we define the potential at infinity $\tilde{\Phi} = \tilde{r}^{-1} \Phi$ then the moments are

$$P_{a_1 \dots a_n} = \tilde{D}_{a_n} P_{a_1 \dots a_{n-1}} = \tilde{D}_{a_1} \dots \tilde{D}_{a_n} \tilde{\Phi} \quad (4)$$

In General Relativity instead of a gravitational field Φ , gravity is described by a metric g_{ab} .

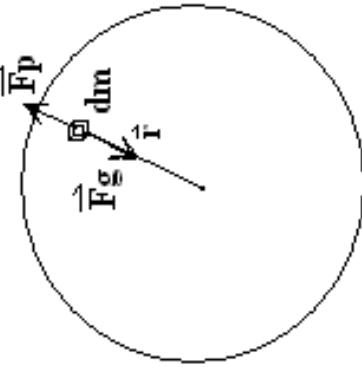
Relativistic multipole moments:

- Generalization of the Newtonian moments,
- Defined for asymptotically flat spacetimes at infinity from a "potential", that is related to the metric, by a recursive relation,
- There are two sets of moments, the Mass moments and the Rotation moments,
- For the two sets of moments we have two generating potentials, Φ_M , Φ_J ,

The multipole moments for stationary and axisymmetric spacetimes can be reduced from tensors to scalars, because of the rotation symmetry.

The moments characterize the structure of the source and of the spacetime.

Compact Objects: Fluid configurations that are in equilibrium by the action of their self-gravity and their internal forces.



Newtonian Stars

Hydrostatic equilibrium (spherical symmetry):

$$\nabla P = -\rho \nabla \Phi \Rightarrow \frac{dP}{dr} = -\frac{d\Phi}{dr} \rho = -G \frac{m(r)}{r^2} \rho$$

$$\text{Mass (spherical symmetry): } \frac{dm}{dr} = 4\pi \rho r^2$$

Field equations: $\nabla^2 \Phi = 4\pi G \rho$, Equation of state for the fluid: $P = P(\rho)$.

A fluid configuration is in equilibrium if the particular configuration minimizes the total energy. Assuming a polytropic equation of state, $P = K \rho^\Gamma \Rightarrow u = K \frac{\rho^{\Gamma-1}}{\Gamma-1}$,

$$E = E_{int} + W = \int u dm - \int \frac{G m dm}{r} = < u > M - a_1 G \frac{M^2}{R} = a_2 K M \rho_c^{\Gamma-1} - a_3 G \rho_c^{1/3} M^{5/3}$$

$$\left. \frac{dE}{d\rho_c} \right|_{M=const.} = 0 \Rightarrow M \propto \rho_c^{(\Gamma-4/3)(3/2)}$$

Relativistic Stars

Instead of a gravitational field Φ , gravity is described by a metric g_{ab} . In spherical symmetry the metric can take the form $ds^2 = -e^{2\Phi} dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$.

Field equations: $G^{ab} = 8\pi G T^{ab}$,

$$\text{Equation for the field } \Phi: \frac{d\Phi}{dr} = \frac{m(r) + 4\pi r^3 P}{r(r-2m(r))},$$

$$\text{Definition of the Mass: } \frac{dm}{dr} = 4\pi \rho r^2,$$

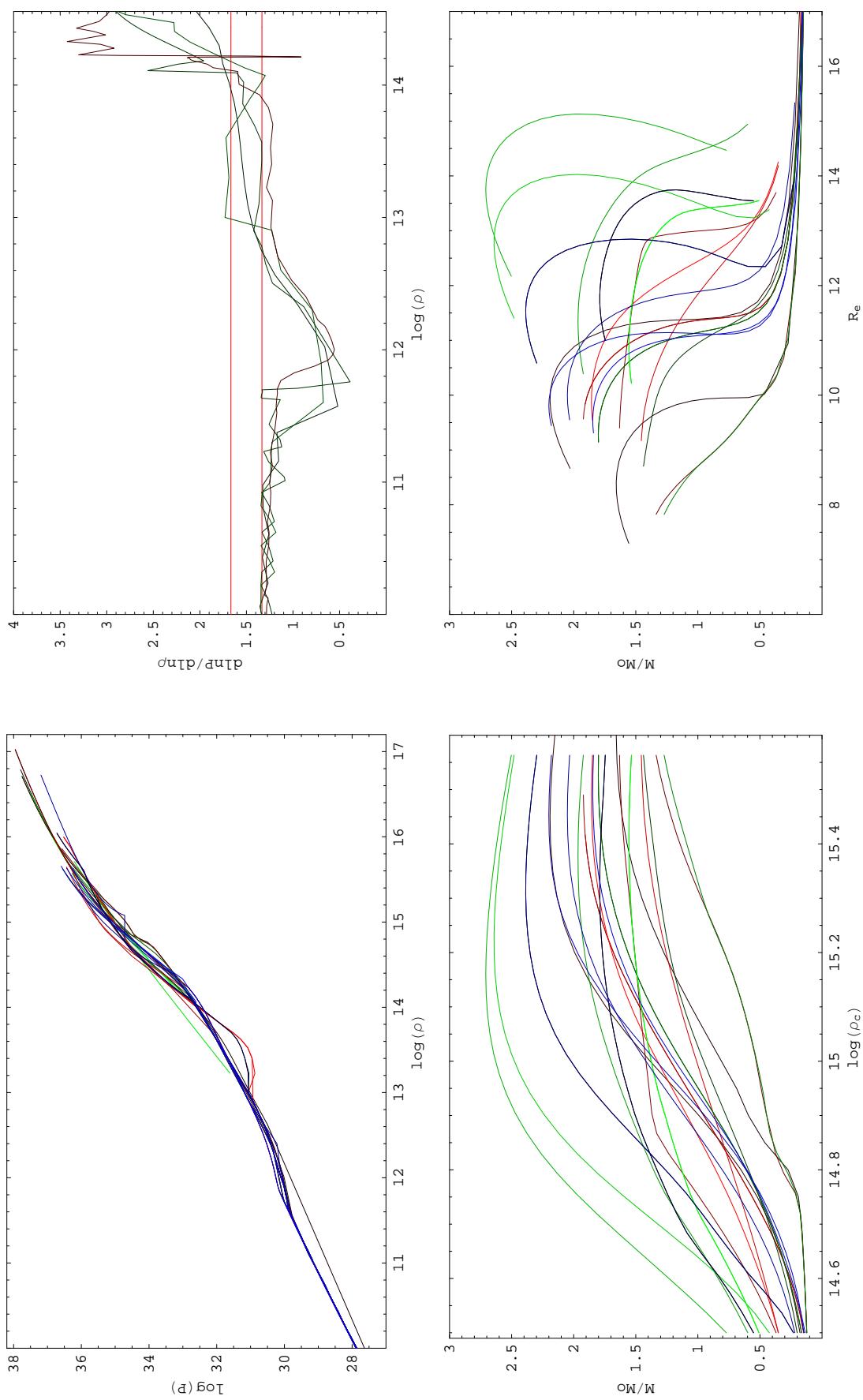
$$\text{Hydrostatic equilibrium: } \frac{dP}{dr} = -(\rho + P) \frac{d\Phi}{dr}.$$

$$\frac{dP}{dr} = -\frac{\rho m(r)}{r^2} \left(1 + \frac{P}{\rho}\right) \left(1 + \frac{4\pi P r^3}{m(r)}\right) \left(1 - \frac{2m(r)}{r}\right)^{-1},$$

Equation of state for the fluid: $P = P(\rho)$.

The spacetime outside the star is the Schwarzschild spacetime:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2.$$



The line element for a stationary and axially symmetric spacetime (the spacetime admits a timelike, ξ^a , and a spacelike, η^a , killing field, i.e. it has rotational symmetry and symmetry in translations in time) is,

$$ds^2 = -e^{2\nu} dt^2 + r^2 \sin^2 \theta B^2 e^{-2\nu} (d\phi - \omega dt)^2 + e^{2(\zeta-\nu)} (dr^2 + r^2 d\theta^2),$$

Field equations in the frame of the ZAMO:

$$\mathbf{D} \cdot (B \mathbf{D}\nu) = \frac{1}{2} r^2 \sin^2 \theta B^3 e^{-4\nu} \mathbf{D}\omega \cdot \mathbf{D}\omega + 4\pi B e^{2\zeta-2\nu} \left[\frac{(\epsilon + p)(1 + u^2)}{1 - u^2} + 2p \right], \quad (5)$$

$$\mathbf{D} \cdot (r^2 \sin^2 \theta B^3 e^{-4\nu} \mathbf{D}\omega) = -16\pi r \sin \theta B^2 e^{2\zeta-4\nu} \frac{(\epsilon + p)u}{1 - u^2}, \quad (6)$$

$$\mathbf{D} \cdot (r \sin \theta \mathbf{D}B) = 16\pi r \sin \theta B e^{2\zeta-2\nu} p, \quad (7)$$

Asymptotic expansion of the metric functions:

$$\nu = \sum_{l=0}^{\infty} \nu_{2l}(r) P_{2l}(\mu), \quad \omega = \sum_{l=0}^{\infty} \omega_{2l}(r) P_{2l+1,\mu}(\mu),$$

$$B = \sum_{l=0}^{\infty} B_{2l}(r) T_{2l}^{1/2}(\mu),$$

where P_l are the Legendre polynomials, $\mu = \cos \theta$, and $T_l^{1/2}$ are the Gegenbauer polynomials.

From the field equations we have the asymptotic expansion:

$$\begin{aligned} \nu &= \left\{ -\frac{M}{r} + \frac{1}{3}\tilde{B}_0\frac{M}{r^3} + \frac{J^2}{r^4} + \left[-\frac{\tilde{B}_0^2}{5} + \frac{\tilde{B}_2^2}{15} - \frac{12J^2}{5} \right] \frac{M}{r^5} + \dots \right\} \\ &\quad + \left\{ \frac{\tilde{\nu}_2}{r^3} - 2\frac{J^2}{r^4} + [\dots] \frac{1}{r^5} + \dots \right\} P_2(\mu) \\ &\quad + \left\{ \frac{\tilde{\nu}_4}{r^5} + \dots \right\} P_4(\mu) + \dots \end{aligned} \tag{8}$$

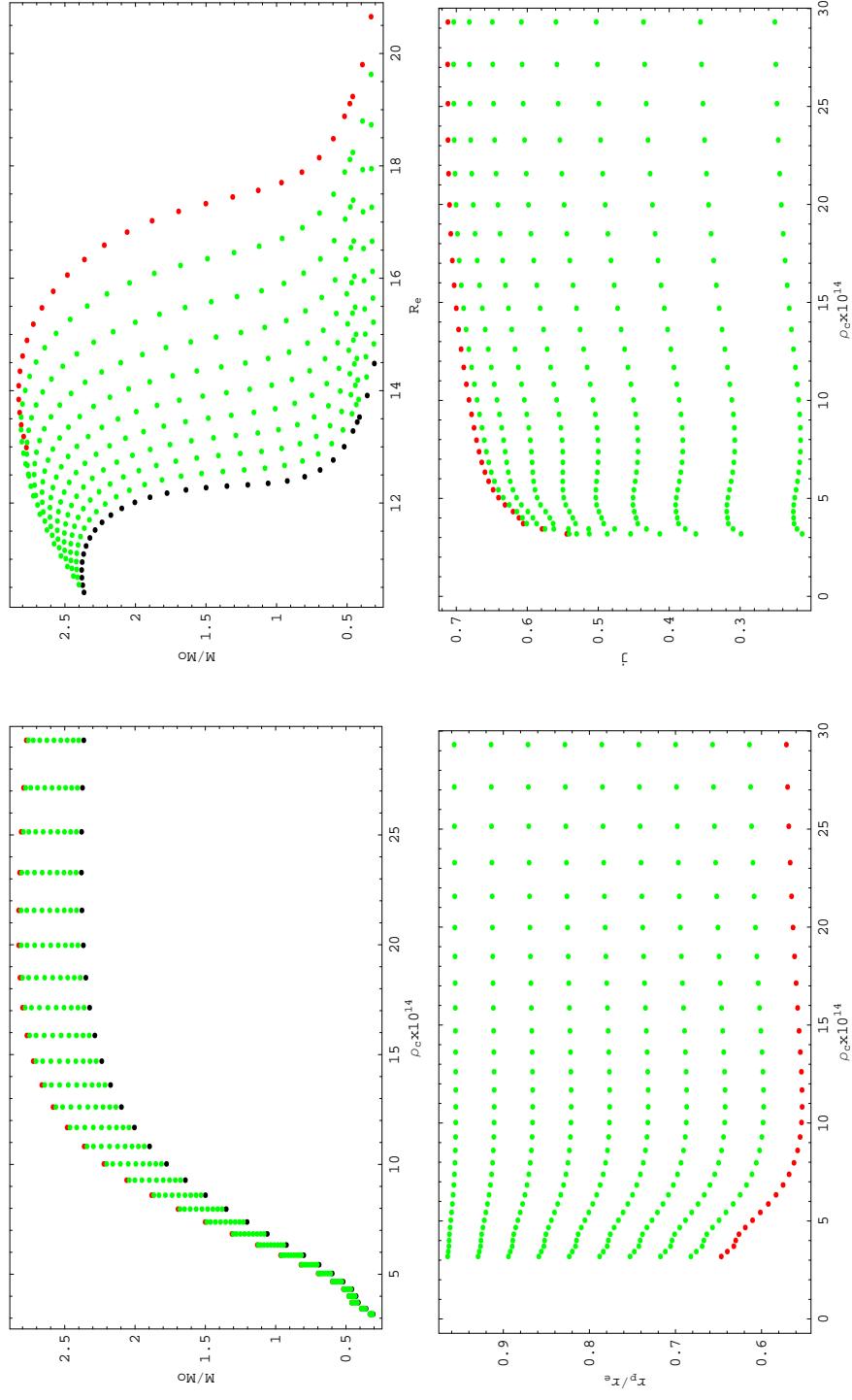
$$\begin{aligned} \omega &= \left\{ \frac{2J}{r^3} - \frac{6JM}{r^4} + \frac{6}{5} \left[8 - 3\frac{\tilde{B}_0}{M^2} \right] \frac{JM^2}{r^5} + (\dots) \frac{J}{r^6} + \dots \right\} P_{1,\mu}(\mu) \\ &\quad + \left\{ \frac{\tilde{\omega}_2}{r^5} + (\dots) \frac{1}{r^6} - \dots \right\} P_{3,\mu}(\mu) + \dots \end{aligned} \tag{9}$$

$$B = \left(\frac{\pi}{2} \right)^{1/2} \left(1 + \frac{\tilde{B}_0}{r^2} \right) T_0^{1/2}(\mu) + \left(\frac{\pi}{2} \right)^{1/2} \frac{\tilde{B}_2}{r^4} T_2^{1/2}(\mu) + \dots \tag{10}$$

The spacetime outside the star to lowest order in $1/r$ is of the form,

$$ds^2 = - \left(1 - \frac{2M}{r} \right) dt^2 + 4\sin^2\theta \frac{J}{r} d\phi dt + \left(1 + \frac{2M}{r} \right) (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi).$$

For the APR equation of state:



The models with the fastest rotation have a spin parameter, $j = J/M^2$, around 0.7 and a ratio of the polar radius over the equatorial radius, r_p/r_e , around 0.56.

In general, we assume that the spacetime around compact objects has symmetry with respect to time translations and with respect to rotations around the an axis of symmetry, which is associated to the axis of rotation of the compact object.

Particle motion in a spacetime with symmetries:

The symmetry in time translations implies the existence of a timelike Killing vector ξ^a and the symmetry in rotations implies the existence of a spacelike Killing vector η^a .

The four-momentum of a particle is defined as $p^a = mu^a = m \frac{dx^a}{d\tau}$.

Symmetry in time translations is associated to an integral of motion, energy E

$$E = -p_a \xi^a = -p_t = -g_{tt} p^t - g_{t\phi} p^\phi = m \left(-g_{tt} \frac{dt}{d\tau} - g_{t\phi} \frac{d\phi}{d\tau} \right) \quad (11)$$

Symmetry in rotations is again associated to an integral of motion, angular momentum L

$$L = p_a \eta^a = p_\phi = g_{t\phi} p^t + g_{\phi\phi} p^\phi = m \left(g_{t\phi} \frac{dt}{d\tau} + g_{\phi\phi} \frac{d\phi}{d\tau} \right) \quad (12)$$

There is one more equation, $p^a p_a = -m^2$, which corresponds to,

$$-1 = g_{tt} \left(\frac{dt}{d\tau} \right)^2 + 2g_{t\phi} \left(\frac{dt}{d\tau} \right) \left(\frac{d\phi}{d\tau} \right) + g_{\phi\phi} \left(\frac{d\phi}{d\tau} \right)^2 + g_{zz} \left(\frac{dz}{d\tau} \right)^2 \quad (13)$$

Circular equatorial orbits:

If we define the angular velocity, $\Omega \equiv \frac{d\phi}{dt}$, equation (13) can take the form,

$$\left(\frac{d\tau}{dt}\right)^2 = -g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2$$

The energy and the angular momentum for the circular orbits are,

$$\tilde{E} \equiv E/m = \frac{-g_{tt} - g_{t\phi}\Omega}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}}, \quad \tilde{L} \equiv L/m = \frac{g_{t\phi} + g_{\phi\phi}\Omega}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}}. \quad (14)$$

From the conditions, $\frac{d\rho}{dt} = 0$, $\frac{d^2\rho}{dt^2} = 0$ and $\frac{dz}{dt} = 0$, and the equations of motion obtained assuming the Lagrangian, $L = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b$, the angular velocity can be calculated to be,

$$\Omega = \frac{-g_{t\phi,\rho} + \sqrt{(g_{t\phi,\rho})^2 - g_{tt,\rho}g_{\phi\phi,\rho}}}{g_{\phi\phi,\rho}}.$$

This is the orbital frequency of a particle in a circular orbit on the equatorial plane.

More general orbits:

Equation (13) can take a more general form in terms of the constants of motion,

$$-g_{\rho\rho} \left(\frac{d\rho}{d\tau} \right)^2 - g_{zz} \left(\frac{dz}{d\tau} \right)^2 = 1 - \frac{\tilde{E}^2 g_{\phi\phi} + 2\tilde{E}\tilde{L}g_{t\phi} + \tilde{L}^2}{(g_{t\phi})^2 - g_{tt}g_{\phi\phi}} = V_{eff}, \quad (15)$$

With equation (15) we can study the general properties of the motion of a particle from the properties of the effective potential.

Small perturbations from circular equatorial orbits:

If we assume small deviations from the circular equatorial orbits of the form, $\rho = \rho_c + \delta\rho$ and $z = \delta z$, then we obtain the perturbed form of (15),

$$-g_{\rho\rho} \left(\frac{d(\delta\rho)}{d\tau} \right)^2 - g_{zz} \left(\frac{d(\delta z)}{d\tau} \right)^2 = \frac{1}{2} \frac{\partial^2 V_{eff}}{\partial \rho^2} (\delta\rho)^2 + \frac{1}{2} \frac{\partial^2 V_{eff}}{\partial z^2} (\delta z)^2,$$

This equation describes two harmonic oscillators with frequencies,

$$\kappa_\rho^2 = \frac{g^{\rho\rho}}{2} \frac{\partial^2 V_{eff}}{\partial \rho^2} \Big|_c, \quad \kappa_z^2 = \frac{g^{zz}}{2} \frac{\partial^2 V_{eff}}{\partial z^2} \Big|_c,$$

The differences of these frequencies from the orbital frequency, $\Omega_a = \Omega - \kappa_a$, define precession frequencies.

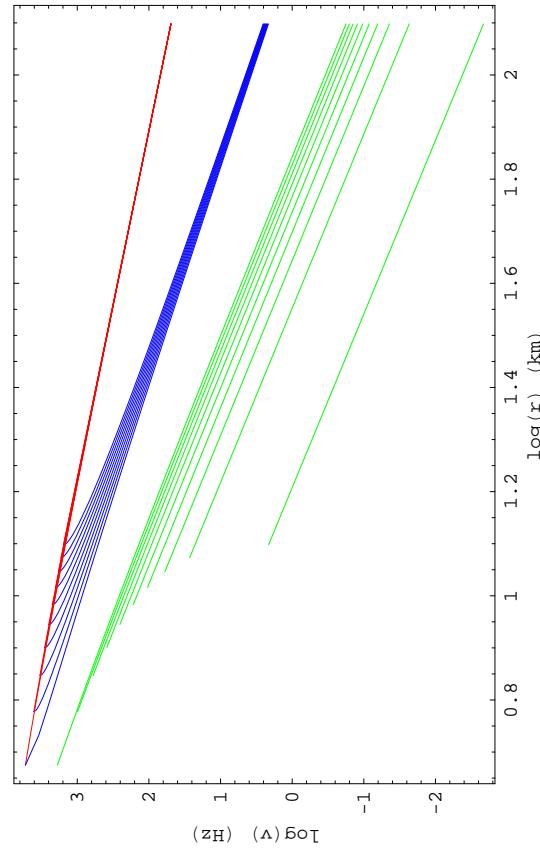
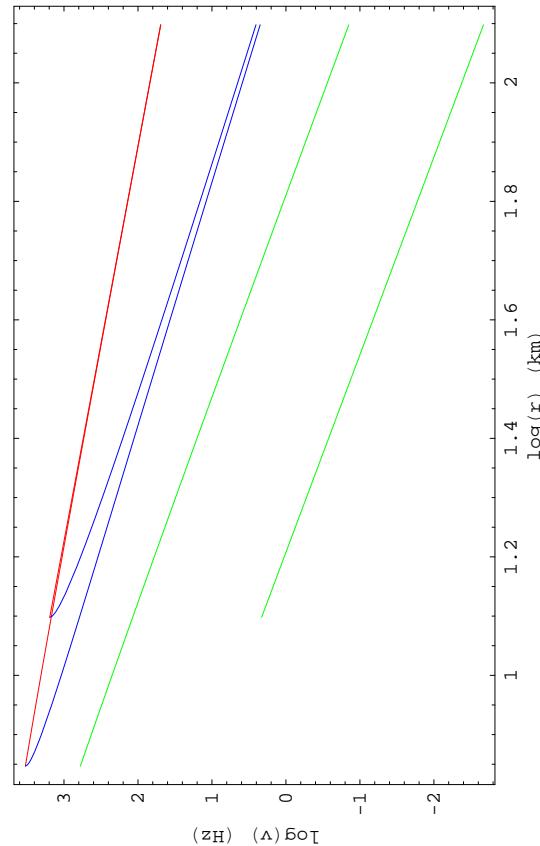
Frequencies for the Kerr spacetime:

$$\text{Orbital frequency (in Hz): } \Omega = \frac{299790\sqrt{M/r^3}}{1+j(M/r)^{3/2}}$$

$$\text{Radial frequency (in Hz): } \kappa_\rho = \Omega (1 - 6(M/r) + 8j(M/r)^{3/2} - 3j^2(M/r)^2)^{1/2}$$

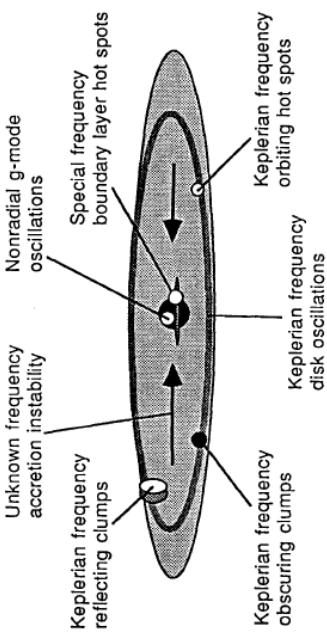
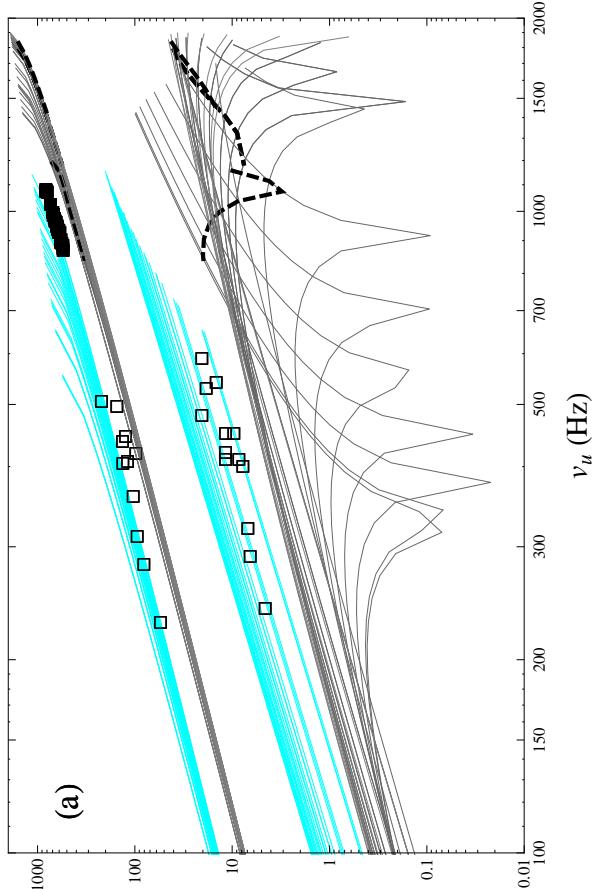
$$\text{Vertical frequency (in Hz): } \kappa_z = \Omega (1 - 4j(M/r)^{3/2} + 3j^2(M/r)^2)^{1/2}$$

Orbital and precession frequencies ($v = \Omega/2\pi$) for an $M = 1.4M_\odot$ Kerr black hole for various j .

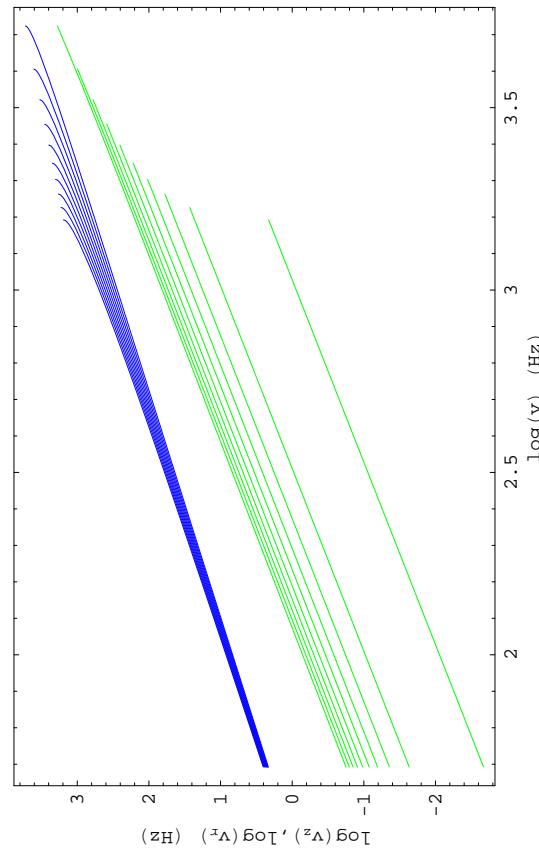


Frequencies for the Kerr spacetime vs Neutron Star spacetime:

Plots of the periastron and the nodal precession frequencies against the orbital frequency,



These frequencies are relevant for QPOs from accretion discs



Returning to the moments, the characteristics of the orbits in a spacetime can be associated to its multipole moments.

The change in the orbital energy per logarithmic frequency interval, $\Delta \tilde{E} = -\frac{d\tilde{E}}{d \ln \Omega}$, for a circular equatorial orbit is,

$$\begin{aligned}\Delta \tilde{E} &= \frac{1}{3}v^2 - \frac{1}{2}v^4 + \frac{20}{9}\frac{S_1}{M^2}v^5 + \left(-\frac{27}{8} + \frac{M_2}{M^3}\right)v^6 + \frac{28}{3}\frac{S_1}{M^2}v^7 \\ &\quad + \left(-\frac{225}{16} + \frac{80}{27}\frac{S_1^2}{M^4} + \frac{70}{9}\frac{M_2}{M^3}\right)v^8 + \left(\frac{81}{2}\frac{S_1}{M^2} + 6\frac{S_1 M_2}{M^5} - 6\frac{S_3}{M^4}\right)v^9 + \dots\end{aligned}$$

where $v = (M\Omega)^{1/3}$ and $S_1 = J$. Can be associated to the spectrum of an accretion disc.

The precession frequencies are also related to the multipole moments,

$$\begin{aligned}\frac{\Omega_\rho}{\Omega} &= 3v^2 - 4\frac{S_1}{M^2}v^3 + \left(\frac{9}{2} - \frac{3M_2}{2M^3}\right)v^4 - 10\frac{S_1}{M^2}v^5 + \left(\frac{27}{2} - 2\frac{S_1^2}{M^4} - \frac{21M_2}{2M^3}\right)v^6 + \dots \\ \frac{\Omega_z}{\Omega} &= 2\frac{S_1}{M^2}v^3 + \frac{3M_2}{2M^3}v^4 + \left(7\frac{S_1^2}{M^4} + 3\frac{M_2}{M^3}\right)v^6 + \left(11\frac{S_1 M_2}{M^5} - 6\frac{S_3}{M^4}\right)v^7 + \dots\end{aligned}$$

If specific QPOs are related to the orbital frequencies, they could be used to measure the multipole moments of the central object.

A theoretical application: Calculating the multipole moments of rapidly rotating neutron stars.

The moments can be calculated from $\Delta \tilde{E}$.

For the spacetime that describes the exterior of neutron stars, with metric functions given by eq. (8-10), $\Delta \tilde{E} = -\frac{d\tilde{E}}{d\log \Omega} = -\frac{v}{3} \frac{d\tilde{E}}{dv}$, will give,

$$\begin{aligned} \Delta \tilde{E} = & \frac{v^2}{3} - \frac{v^4}{2} + \frac{20jv^5}{9} - \frac{(89 + 32b + 24q)}{24}v^6 + \frac{28jv^7}{3} \\ & - \frac{5(1439 + 896b - 256j^2 + 672q)v^8}{432} \\ & + \frac{((421 + 64b - 60q)j - 90w_2)v^9}{10} + O(v^{10}), \end{aligned} \quad (16)$$

$$\text{where } j = \frac{J}{M^2}, \quad q = \frac{\tilde{\nu}_2}{M^3}, \quad w_2 = \frac{\tilde{\omega}_2}{M^4}, \quad b = \frac{B_0}{M^2}.$$

The moments are evaluated to be,

$$M_2 = -\tilde{\nu}_2 - \frac{4}{3}(\frac{1}{4} + b)M^3, \quad (17)$$

$$S_3 = \frac{3}{2}\tilde{\omega}_2 - \frac{12}{5}(\frac{1}{4} + b)jM^4. \quad (18)$$